① For minimum-fuel problem, how to quickly get u^*?

$|u^*| + p^T B u^* \le |u| + p^T B u$ (For $p^T B$ is $1 \times m$ vector, each u^* is independent & same)

let: $f(x) = |x| + \lambda x$, $\lambda = p^T B$, $x \in [-1, 1]$, we want to get the $u^* = \arg\min_{|x| \le 1} f(x) = x^*$

$$f'(x) = \begin{cases} -1 + \lambda, & -1 \le x < 0^- \\ 1 + \lambda, & 1 \ge x > 0^+ \end{cases}$$

when $\lambda \le -1$, $f(x) \downarrow$, $f(x)_{min} = f(x^* = 1)$

$\lambda \ge 1$, $f(x) \uparrow$, $f(x)_{min} = f(x^* = -1)$

$-1 < \lambda < 1$, $f(x)$ first $\downarrow$ then $\uparrow$, $f(x)_{min} = f(x^* = 0)$

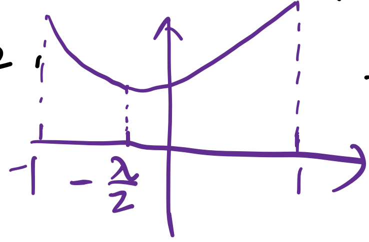
For min-energy problem, in this way can also easily get u^*,

$\int u^2 \, dt$

let: $f(x) = x^2 + \lambda x$

$f'(x) = 2x + \lambda$, $x \in [-1, 1]$, when $\lambda \le -2$, $f(x) \downarrow$, $f(x)_{min} = f(x^* = 1)$

$\lambda \ge 2$, $f(x) \uparrow$, $f(x)_{min} = f(x^* = -1)$

$-2 < \lambda < 2$,  $f(x)_{min} = f(x^* = -\frac{\lambda}{2})$

② prove: In min-time problem, $p^T B \neq 0$ for $t \in [t_1, t_2] = S$

(also consider the $p \& B \neq 0$ condition)

If for $t \in S$, $p^T B = 0$, because $H = 1 + p^T(Ax + Bu)$

so, $\dot{p} = \frac{-\partial H}{\partial x} = -A^T p \Rightarrow (\dot{p})^T = -p^T A$

And, B is a constant matrix, so $\frac{d(p^T B)}{dt} = \dot{p}^T B = -p^T A B$

Since $p^T B = 0$, so $\frac{d(p^T B)}{dt} = 0 = -p^T A B$

And, $\frac{d^2(p^T B)}{dt^2} = -\dot{p}^T A B = p^T A^2 B$ (similarly $p^T A^2 B = 0$)

So, for any $k \geq 0$, $\frac{d^k(p^T B)}{dt^k} = (-1)^k p^T A^k B = 0 \Rightarrow p^T A^k B = 0$

That is: $\underset{t_m \in S}{p^T(t_m)} \underbrace{[B, AB, \dots A^{n-1}B]}_{\text{controllability matrix!}} = \vec{0}$ ---- ①

If (A, B) is controllable, this should be full-rank, so according to Linear Algebra, the only solution of ① is $p(t_m) = 0$ for any $t_m \in S$, but $\underbrace{\text{it's impossible!}}$

$$H = 1 + p^T A x + p^T B u = 0, \quad \text{for } t \in S$$

$H(p^T(t) = 0) = 1 \neq 0$

Q.E.D.